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Bäcklund transformation for the non-isospectral and variable-coefficient non-linear Schrödinger equation

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Abstract. A non-linear Schrödinger equation (NLSE) with a non-isospectral Lax pair and variable coefficients is studied. This equation is shown to be an integrability condition for an AKNS system, a Γ-Riccati system and a Lax pair. These three systems are proved to be equivalent. They are all invariant under some generalisation of the Konno-Wadati transformation. Related to this (NLSE) there is also a variable-coefficients evolution equation for the function Γ, derived from the Γ-Riccati system. A Bäcklund transformation for this Γ equation has been constructed. With this, a new Bäcklund transformation for the (NLSE) is presented so that successive generation of new solutions is reduced to integration only.

1. Introduction

The purpose of this paper is to study a generalisation of the well known soliton equation, the non-linear Schrödinger equation (NLSE) [1-3]

$$\mathrm{i}q_t + q_{xx} + 2q^2q^* = 0. \tag{1.1}$$

To be more specific, we shall consider the following evolution equation for $q(x, t)$:

$$q_t = (k_0/2)(q_{xx} + 2q^2q^*) + k_1q_x + (2k_2 + h_1)q \tag{1.2}$$

where k_0, k_1, k_2 and h_1 are some functions of x and t. For $k_0 = 2\mathrm{i}$, $k_1 = k_2 = h_1 = 0$, it reduces to the NLSE (1.1). Thus it is a variable-coefficient NLSE and, as we shall see, it is also non-isospectral. We call it the generalised non-linear Schrödinger equation (GNLSE). This equation has been treated by Calogero and Degasperis [4, § 6.2] using their method of generalised Wronskian techniques and by Newell [5]. One may also consult [6, 7] for further details. From another point of view, Riccati equations for the ratios of components of the wavefunction of the NLSE are derived from the linear equations for the wavefunctions, and the Bäcklund transformations (BT) are obtained from suitable automorphisms of these Riccati equations. This approach has been developed by Kono and Wadati [8], Chen [9] and Fordy [10], among others. In [8] the automorphism introduced is $\Gamma \to 1/\Gamma^*$. One of the purposes of the present paper is to demonstrate that the Riccati equations for Γ (one in x and one in t) are both invariant under certain transformations induced by the above automorphism for the GNLSE. Furthermore, we shall construct a new BT for the GNLSE which can be considered as an extension and improvement of the one given in [8]. There, a BT was given for the constant-coefficient and isospectral NLSE. However, that BT has a drawback in that, if we apply it twice, it gives us back the original solution so that successive generation of new solutions is impossible. Our new BT for the GNLSE will successively generate a hierarchy of solutions via integration only, overcoming the difficulty inherent in the BT in [8]. It is of interest to note that the time evolution equation for Γ can be expressed in terms of Γ and its derivatives only, and the BT for the GNLSE is closely related to the solution of this Γ evolution equation. Therefore, a BT for this equation will first be presented before constructing the new BT for the GNLSE.

2. A generalised NLSE

Consider the general AKNS system

$$\psi_{1x} = \eta\psi_1 + q\psi_2 \tag{2.1}$$

$$\psi_{2x} = r\psi_1 - \eta\psi_2 \tag{2.2}$$

$$\psi_{1t} = A(x, t, \eta)\psi_1 + B(x, t, \eta)\psi_2 \tag{2.3}$$

$$\psi_{2t} = C(x, t, \eta)\psi_1 - A(x, t, \eta)\psi_2 \tag{2.4}$$

where η is a complex function of t satisfying the following linear-type equation (non-isospectral condition):

$$\eta_t = h_1(t)\eta + h_2(t) \tag{2.5}$$

where $h_1(t)$ and $h_2(t)$ are some known complex functions of t.

The integrability conditions of equations (2.1)-(2.4) are

$$\eta_t - A_x + qC - rB = 0 \tag{2.6}$$

$$q_t - B_x + 2\eta B - 2qA = 0 \tag{2.7}$$

$$r_t - C_x + 2rA - 2\eta C = 0. \tag{2.8}$$

We choose $r = -q^*$, and A, B and C in (2.6)-(2.8) to be the following polynomials in η:

$$A = \tfrac{1}{2}k_0(qq^* + 2\eta^2) + k_1\eta + k_2 \tag{2.9}$$

$$B = \tfrac{1}{2}k_0(q_x + 2\eta q) + k_1 q \tag{2.10}$$

$$C = \tfrac{1}{2}k_0(q_x^* - 2\eta q^*) - k_1 q^* \tag{2.11}$$

$$k_j = k_j(x, t) = h_j(t)x + l_j(t) \qquad j = 1, 2 \tag{2.12}$$

where $l_j(t)$ are some arbitrary complex functions of t. Then (2.6)-(2.8) gives the equation

$$q_t = \tfrac{1}{2}k_0(q_{xx} + 2q^2 q^*) + k_1 q_x + (2k_2 + h_1)q. \tag{2.13}$$

This is just the GNLSE (1.2). It is known that the following theorem holds.

Theorem 1. The AKNS system (2.1)-(2.5), with (2.9)-(2.12), is integrable if and only if the functions q satisfy equation (2.13).

3. Γ-Riccati equation system

For convenience, we will transform the AKNS system (2.1)-(2.4) into an equivalent system, called the Γ-Riccati equation system (or Γ system).

Following [8], we introduce a new function

$$\Gamma=\psi_1/\psi_2. \tag{3.1}$$

Taking derivatives of (3.1) with respect to x and t respectively and using (2.1)-(2.4), we get the following Γ system:

$$\Gamma_x=2\eta\Gamma+q-r\Gamma^2 \tag{3.2}$$

$$\Gamma_t=2A\Gamma+B-C\Gamma^2. \tag{3.3}$$

By directly verifying the equality $\Gamma_{xt}=\Gamma_{tx}$, one finds that (3.2) and (3.3) have the same integrability conditions (2.6)-(2.8) as the AKNS system (2.1)-(2.4).

Now, suppose that equations (3.2) and (3.3), with $r=-q^*$, are integrable, then q, A, B and C can be determined by (2.13) and (2.9)-(2.11), respectively. We may apply these quantities to the Γ system (3.2) and (3.3) and solve it for Γ; then we define two functions ψ_1 and ψ_2 by the two equations

$$(\psi_{2x}+\eta\psi_2)/r\psi_2=\Gamma \tag{3.4}$$

$$\psi_1=(\psi_{2x}+\eta\psi_2)/r. \tag{3.5}$$

Hence we get

$$\psi_2=\psi_2^0\exp\left(\int_{x_0}^{x}(r\Gamma-\eta)\,\mathrm{d}x\right) \tag{3.6}$$

$$\Gamma=\psi_1/\psi_2 \tag{3.7}$$

$$\psi_{2x}=r\psi_1-\eta\psi_2 \tag{3.8}$$

where $\psi_2^0=\psi_2(x_0,t)$, and will be determined later. Substituting (3.7) into (3.2) and using (3.8), we obtain the equation

$$\psi_{1x}=\eta\psi_1+q\psi_2. \tag{3.9}$$

Differentiating (3.6) with respect to t and applying (3.3) and (3.7) gives

$$\psi_{2t}=C\psi_1-A\psi_2+[(\psi_{2t}^0/\psi_2^0)-(C\Gamma-A)_{x=x_0}]\psi_2. \tag{3.10}$$

We now choose ψ_2^0 to satisfy

$$\psi_{2t}^0/\psi_2^0=(C\Gamma-A)_{x=x_0} \tag{3.11}$$

and (3.10) then leads to

$$\psi_{2t}=C\psi_1-A\psi_2. \tag{3.12}$$

Substituting (3.7) into (3.3) and applying (3.12), we get another equation

$$\psi_{1t}=A\psi_1+B\psi_2. \tag{3.13}$$

Equations (3.8), (3.9), (3.12) and (3.13) are nothing other than the AKNS system (2.1)-(2.4).

The above discussion means that the AKNS system (2.1)-(2.4) and the Γ system (3.2) and (3.3) are equivalent under the transformation (3.1), (3.5) and (3.6). We encapsulate this result in the following theorem.

Theorem 2. The AKNS system (2.1) and (2.4) and the Γ system (3.2) and (3.3) are equivalent under the transformation (3.1), (3.5) and (3.6).

Substituting $r=-q^*$ into (3.2) and (2.9)-(2.11) into (3.3), we get

$$\Gamma_x = 2\eta\Gamma + q + q^*\Gamma^2 \tag{3.14}$$

$$\Gamma_t = \tfrac{1}{2}k_0\{\Gamma_{xx} + 2\Gamma[qq^* - (q^*\Gamma)_x]\} + k_1\Gamma_x + 2k_2\Gamma. \tag{3.15}$$

By theorem 2, this Γ system possesses the same integrability condition GNLSE (2.13) as the following AKNS system:

$$\psi_{1x} = \eta\psi_1 + q\psi_2 \tag{3.16}$$

$$\psi_{2x} = -q^*\psi_1 - \eta\psi_2 \tag{3.17}$$

$$\psi_{1t} = A(x,t,\eta)\psi_1 + B(x,t,\eta)\psi_2 \tag{3.18}$$

$$\psi_{2t} = C(x,t,\eta)\psi_1 - A(x,t,\eta)\psi_2. \tag{3.19}$$

We will use this Γ system to study the problem of solving the GNLSE (2.13).

4. Invariance of the Γ system

Solving equation (3.14) for q, we get

$$q = \frac{\Gamma_x - \Gamma^2\Gamma_x^*}{1-|\Gamma|^4} - \frac{2\Gamma}{1-|\Gamma|^4}(\eta - \eta^*|\Gamma|^2). \tag{4.1}$$

Let

$$\Gamma' = 1/\Gamma^* \tag{4.2}$$

and substitute this Γ′ into (4.1) to obtain a new q, denoted by q'

$$q' = -q - \sigma \qquad \sigma = \frac{2\Gamma}{1+|\Gamma|^2}(\eta + \eta^*). \tag{4.3}$$

In [8], it was pointed out that (3.14) is invariant under the transformation (4.2) and (4.3), we now show that (3.15) is also invariant under these transformations.

Substituting (4.2) and (4.3) into (3.15), we have

$$\begin{aligned}-(\Gamma^*)^{-2}\Gamma_t^* = \tfrac{1}{2}k_0\{&-(\Gamma^*)^{-2}\Gamma_{xx}^* + 2(\Gamma^*)^{-3}(\Gamma_x^*)^2 \\ &+ (2/\Gamma^*)[qq^* + q\sigma^* + q^*\sigma + \sigma\sigma^* + (q^*/\Gamma^* + \sigma^*/\Gamma^*)_x]\} \\ &- k_1(\Gamma^*)^{-2}\Gamma_x^* + 2k_2/\Gamma^*\end{aligned}$$

or, multiplying the equation by $(-\Gamma^*)^{-2}$ and taking the complex conjugate,

$$\Gamma_t = -\tfrac{1}{2}k_0^*\{\Gamma_{xx} + 2\Gamma[qq^* - (q^*\Gamma)_x + \rho]\} + k_1^*\Gamma_x - 2k_2^*\Gamma \tag{4.4}$$

where

$$\begin{aligned}\rho &= -\left(\frac{\Gamma_x}{\Gamma}\right)_x + \left(\frac{q+\sigma}{\Gamma}\right)_x + (q^*\Gamma)_x + q^*\sigma + q\sigma^* + \sigma\sigma^* \\ &= [\Gamma^{-1}(-\Gamma_x + q + q^*\Gamma^2 + \sigma)]_x + q^*\sigma + q\sigma^* + \sigma\sigma^*.\end{aligned} \tag{4.5}$$

We will show that the quantity ρ in (4.5) is null.

Substituting (3.14) into (4.5) gives

$$\rho = (\alpha/\Gamma)_x + q^*\sigma + q\sigma^* + \sigma\sigma^*. \tag{4.6}$$

By (4.3) and (3.14), we have

$$\begin{aligned}(\sigma/\Gamma)_x &= -2(\eta+\eta^*)(\Gamma_x\Gamma^* + \Gamma\Gamma_x^*)/(1+\Gamma\Gamma^*)^2 \\ &= -2(\eta+\eta^*)[(q^*\Gamma + q\Gamma^*)(1+\Gamma\Gamma^*) + 2(\eta+\eta^*)\Gamma\Gamma^*]/(1+\Gamma\Gamma^*)^2 \\ &= -(q^*\sigma + q\sigma^* + \sigma\sigma^*).\end{aligned} \tag{4.7}$$

Substitution of (4.7) into (4.6) leads to

$$\rho = 0. \tag{4.8}$$

From now on, we assume that k_0, k_1 and k_2 possess the properties

$$k_0^* = -k_0 \qquad k_1^* = k_1 \qquad k_2^* = -k_2. \tag{4.9}$$

Under condition (4.9) and result (4.8), we see that (4.4) coincides with (3.15). Thus, we have arrived at the following theorem.

Theorem 3. The Γ system (3.14) and (3.15) is invariant under the transformation (4.2) and (4.3).

From theorems 1-3, we obtain a fourth theorem.

Theorem 4. Assume that $q = q(x, t)$ is a solution of the GNLSE (2.13) and Γ is a solution of the Γ system (3.14) and (3.15), then the function $q' = q'(x, t)$ determined by (4.3) is also a solution of the GNLSE (2.13).

Equations (4.2) and (4.3) comprise a BT for the NLSE (1.1) proposed in [8]. We know that it has a drawback in practice as mentioned in the introduction. We will now construct a new BT to make up for this deficiency.

5. Lax pair corresponding to the GNLSE

For the purpose of constructing a BT for the GNLSE, we need to construct a Lax pair corresponding to the GNLSE and to prove the equivalence between this Lax pair and the AKNS system (3.16)-(3.19).

Solving for ψ_1 from (3.17), we get

$$\psi_1 = -(\psi_{2x} + \eta\psi_2)/q^* \tag{5.1}$$

and substituting (5.1) into (3.16) we obtain the Schrödinger equation

$$\psi_{2xx} - (q_x^*/q^*)\psi_{2x} + (qq^* - \eta q_x^*/q^*)\psi_2 = \eta^2\psi_2 \tag{5.2}$$

while substituting (5.1) into (3.19) gives

$$\psi_{2t} = -(A + \eta C/q^*)\psi_2 - (C/q^*)\psi_{2x} \tag{5.3}$$

where A and C are defined by (2.9) and (2.11), respectively. By directly verifying the integrability condition $\psi_{2xxt} = \psi_{2txx}$ of (5.2) and (5.3), and defining

$$B = (A_x - \eta_t - qC)/q^* \tag{5.4}$$

or, equivalently,

$$\eta_t - A_x + qC + q^*B = 0 \tag{5.5}$$

we obtain the two equalities

$$[(q_t^* + C_x + 2q^*A + 2\eta C)/q^*]_x = 0 \tag{5.6}$$

$$q(q_t^* + C_x) + q^*(q_t - B_x) - 2\eta\eta_t - [\eta(q_t^* + C_x + 2\eta C)/q^*]_x = 0. \tag{5.7}$$

By taking the integration constant in (5.6) to be zero, we get

$$q_t^* + C_x + 2q^*A + 2\pi C = 0. \tag{5.8}$$

Applying (5.8) to (5.7) gives

$$q_t - B_x + 2\eta B - 2qA = 0. \tag{5.9}$$

Equations (5.5), (5.8) and (5.9) are the integrability conditions of (5.2) and (5.3). They coincide with (2.6)-(2.8) with $r = -q^*$ in the latter. Therefore, (5.2) and (5.3) possess the GNLSE (2.13) as an integrability condition. We call (5.2) and (5.3) the Lax pair corresponding to the GNLSE (2.13).

We now show that the AKNS system (3.16)-(3.19) can be derived from the Lax pair (5.2) and (5.3). Assuming that (5.2) and (5.3) are integrable, then the quantities q, A, B and C can be determined by (2.13) and (2.9)-(2.11). Letting ψ_2 be a solution of (5.2) and (5.3) and defining

$$\psi_1 = -(\psi_2 2_x + \eta\psi_2)/q^* \tag{5.10}$$

then we have

$$\psi_{2x} = -q^*\psi_1 - \eta\psi_2. \tag{5.11}$$

Taking the derivative of (5.10) with respect to x and using (5.2), we get

$$\psi_{1x} = \eta\psi_1 + q\psi_2. \tag{5.12}$$

Substituting (5.11) into (5.3) gives

$$\psi_{2t} = C\psi_1 - A\psi_2. \tag{5.13}$$

Taking the derivative of (5.10) with respect to t and using (5.4), we obtain

$$\psi_{1t} = A\psi_1 + B\psi_2. \tag{5.14}$$

Thus we have derived the AKNS system (3.16)-(3.19) from the Lax pair (5.2) and (5.3). We state this result in another theorem.

Theorem 5. The Lax pair (5.2) and (5.3) and the AKNS system (3.16)-(3.19) are equivalent under the transformation (5.1).

Theorems 2 and 5 yield a sixth theorem.

Theorem 6. The Lax pair (5.2)-(5.3) and the Γ system (3.14) and (3.15) are equivalent under the transformation

$$\Gamma = -(\psi_{2x} + \eta\psi_2)/(q^*\psi_2) \qquad (= \psi_1/\psi_2) \tag{5.15}$$

or

$$\psi_2 = \psi_2^0 \exp\left(-\int_{x_0}^{x} (q^*\Gamma + \eta)\, \mathrm{d}x \right) \tag{5.16}$$

where ψ_2^0 satisfies the condition (3.11).

6. BT for the Γ evolution equation and the GNLSE

In this section, we will construct a BT for the GNLSE (2.13) from the Γ system (3.14) and (3.15) and the Lax pair (5.2) and (5.3).

Let $q = q(x, t)$ be a known solution of the GNLSE (2.13) and Γ be a corresponding solution of the Γ system (3.14) and (3.15); then by theorem 6 the Lax pair (5.2) and (5.3) has a solution ψ_2 defined by (5.16). Substituting the corresponding q' and Γ' defined by (4.2) and (4.3) into (5.16), we get a new ψ_2, denoted by ψ_2'

$$\psi_2' = \psi_2'^0 \exp\left[-\int_0^x (q'^*\Gamma' + \eta)\,\mathrm{d}x\right] \tag{6.1}$$

where $\psi_2'^0$ is a function of t satisfying the equation (condition (3.11))

$$\psi_{2t}'^0 = \psi_2'^0(C'\Gamma' - A')_{x=x}$$

or

$$\psi_2'^0 = \psi_2'(x_0, t_0)\int_t^t \exp(C'\Gamma' - A')_{x=x_0}\,\mathrm{d}t \tag{6.2}$$

and A' and C' are obtained from (2.9) and (2.11) by replacing q by q'. From theorems 3 and 6 we then have the following.

Theorem 7. The Lax pair (5.2) and (5.3) is invariant under the transformations (4.3) and (6.1), i.e. the following two equalities hold:

$$\psi_{2xx}' - (q_x'^*/q'^*)\psi_{2x}' + (q'q'^* - \eta q_x'^*/q'^*)\psi_2' = \eta^2\psi_2' \tag{6.3}$$

$$\psi_{2t}' = -(A' + \eta C'/q'^*)\psi_2' - (C'/q'^*)\psi_{2x}'. \tag{6.4}$$

By this theorem, the Schrödinger equation (6.3) has a solution (6.1), but then, according to a well known property of second-order linear differential equations, (6.3) will possess a general solution

$$\hat{\psi}_2 = (C_1\mu + C_2)\psi_2' \tag{6.5}$$

where

$$\mu = \int_{x_0}^x q'^*\nu\,\mathrm{d}x \tag{6.6}$$

$$\nu = \exp\left(2\int_{x_0}^x (q'^*\Gamma' + \eta)\,\mathrm{d}x\right) \tag{6.7}$$

and C_1 and C_2 are functions of t. We will show that C_1 and C_2 can be suitably chosen such that (6.5) satisfies (6.4).

Substituting (6.5) into (6.4) and recalling that ψ_2' satisfies (6.4), we get

$$C_{1t}\mu + C_{2t} = -C_1(\mu_t + C'\nu). \tag{6.8}$$

Substituting (6.1) into (6.4), cancelling the exponentials on both sides of the equality and then taking the derivative with respect to x, we obtain the conservation law

$$(q'^*\Gamma' + \eta)_t + (C'\Gamma' - A')_x = 0. \tag{6.9}$$

Taking the derivative of (6.6) with respect to t and using (6.9), we have

$$\mu_t = \int_{x_0}^{x} (q_t'^* \nu - 2(C'\Gamma' - A')|_{x_0}^{x} q'^* \nu)\, dx$$

$$= \int_{x_0}^{x} q_t'^* \nu \, dx - 2\int_{x_0}^{x} (C'\Gamma' - A') q'^* \nu \, dx + 2(C'\Gamma' - A')_{x=x_0}\mu. \tag{6.10}$$

Noting that q', A', B' and C' satisfy (2.13) and (2.9)-(2.11), integrating by parts and using (4.9) we have

$$\int_{x_0}^{x} q_t'^* \nu \, dx = -C'\nu|_{x_0}^{x} + 2\int_{x_0}^{x} q'^*(C'\Gamma' - A')\nu \, dx. \tag{6.11}$$

Substituting (6.11) into (6.10) gives

$$\mu_t = -C'\nu + C'_{x=x_0} + 2(C'\Gamma' - A')_{x=x_0}\mu \tag{6.12}$$

and substituting (6.10) into (6.8) leads to

$$C_{1t}\mu + C_{2t} = -C_1[C'_{x=x_0} + 2(C'\Gamma' - A')_{x=x_0}\mu]. \tag{6.13}$$

Comparing the two sides of (6.13), we obtain the following ordinary differential equations for C_1 and C_2:

$$C_{1t} = -2(C'\Gamma' - A')_{x=x_0} C_1 \tag{6.14}$$

$$C_{2t} = -C'_{x=x_0} C_1. \tag{6.15}$$

Solving (6.14) and (6.15) and using (6.2), we find that

$$C_1 = \alpha_1 (\psi_2'^0)^{-2} \tag{6.16}$$

$$C_2 = -\alpha_1 \int_{t_0}^{t} (C'\psi_2'^{-2})_{x=x_0}\, dt + \alpha_2 \tag{6.17}$$

where α_1 and α_2 are arbitrary constants.

Therefore, if we take the two functions C_1 and C_2 in (6.5) to be (6.16) and (6.17), then (6.5), which satisfies (6.3), will also satisfy equation (6.4). This actually constitutes a proof of the following theorem.

Theorem 8. The Lax pair (5.2) and (5.3) is invariant under the transformations (4.3) and (6.5), where the ψ_2' in (6.5) is determined by (6.1), and C_1 and C_2 are determined by (6.16) and (6.17).

Now substitute (6.5) and (4.3) into (5.15) to obtain a new Γ, denoted by $\hat{\Gamma}$:

$$\hat{\Gamma} = \frac{1}{\Gamma^*} - C_1 \exp\left[2\int_{x_0}^{x}\left(\frac{q'^*}{\Gamma^*} + \eta\right) dx\right]$$

$$\times \left\{ C_1 \int_{x_0}^{x} q'^* \exp\left[2\int_{x_0}^{x}\left(\frac{q'^*}{\Gamma^*} + \eta\right) dx\right] dx + C_2 \right\}^{-1} \tag{6.18}$$

where we have made the substitution (4.2) on the right-hand side. Thus, from theorems 3, 4, 6 and 8, we have the following.

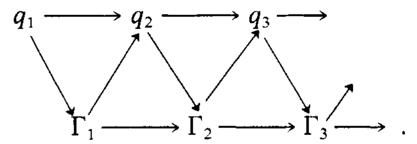
Theorem 9. If Γ is a solution of the Γ system (3.14) and (3.15) corresponding to q, which is a solution of the GNLSE (2.13), then the function $\hat{\Gamma}$, defined by (6.18), is also a solution of (3.14) and (3.15), corresponding to q', where q' is determined by Γ in (4.3) and is another solution of the GNLSE (2.13).

This theorem indicates that (4.3) is a BT for the GNLSE (2.13), and (6.18) is a BT for the solution Γ of the Γ system (3.14) and (3.15). Note that when (4.1), derived from (3.14), is substituted into (3.15), q is eliminated and an evolution equation for Γ results. Thus, (6.18) is actually on auto-BT for (3.15). From these BT, starting from an initial solution q_1 (seed solution) of (2.13), we can obtain two hierarchies of infinitely many solutions of (2.13) and (3.14) and (3.15) without solving any differential equation except that for Γ_1. Denoting the two hierarchies of solutions by

$$q_1, q_2, q_3, \ldots \tag{6.19}$$

$$\Gamma_1, \Gamma_2, \Gamma_3, \ldots \tag{6.20}$$

then the procedure of obtaining these solutions can be depicted in the diagram



(6.21)

The transformation formula for Γ in (6.18) is a generalisation of the Konno–Wadati formula (4.2), since if $C_1 = 0$ then (6.18) reduces to (4.2). From the discussions above we see that the role which the Γ equation (3.15) and (4.1) plays is very similar to the η^2-dependent modified Korteweg-de Vries equation in relation to the KdV equation [11-13]. Hence, it may well be called the η-dependent modified GNLSE.

References

[1] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)

[2] Novikov S, Manakov S V, Pitaevski L P and Zakharov V E 1984 *Theorey of Solitons, The Inverse Scattering Method* (New York: Consultants Bureau)

[3] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)

[4] Calogero F and Degasperis A 1982 *Spectral Transforms and Solitons* (Amsterdam: North-Holland)

[5] Newell A C 1978 *Nonlinear Evolution Equation Sovable by the Inverse Spectral Transform* (*Research Notes in Mathematics* **26**) ed F Calogero, (London: Pitman) pp 127–79

[6] Calogero F and Degasperis A 1978 *Lett. Nuovo Cimento* **22** 131–7

[7] Calogero F and Degasperis A 1978 *Lett. Nuovo Cimento* **22** 420–4

[8] Konno K and Wadati M 1975 *Prog. Theor. Phys.* **53** 1652–6

[9] Chen H 1976 *Bäcklund Transformation* (*Lecture Notes in Mathematics* **515**) ed R Miura (Berlin: Springer) pp 241–51

[10] Fordy A P 1983 *Proc. R. Irish Acad.* A **83** 75–93

[11] Chan W L and Zheng Y-k 1987 *Lett. Math. Phys.* **14** 293–301

[12] Jager E M and Spannenburg S 1985 *J. Phys. A: Math. Gen.* **18** 2177–89

[13] Zheng Y and Chan W L 1988 *J. Math. Phys.* **29** 308–14